

**PRACTICE QUESTIONS ON BAYES'S FORMULA AND ON
PROBABILITY
(NOT TO BE HANDED IN)**

1. REMARKS

If you find any errors in this document, please alert me.

Remark 1. First, I'll make a remark about question 40 from section 12.4 in the book. Let A = event that first card is a spade and B =event that second card is a spade. As part of this question, you computed (presumably using the total law of probability) that

$$P(B) = P(A)P(B | A) + P(A^c)P(B | A^c) = \frac{13}{52} \times \frac{12}{51} + \frac{39}{52} \times \frac{13}{51} = \frac{1}{4}.$$

Note that in this case, of course, you already knew actually that

$$P(B) = \frac{13}{52} = \frac{1}{4},$$

since there are 13 spades in 52 cards, therefore the *unconditional* probability of B is $\frac{13}{52}$.

The law of total of total probability gives you a method for computing the unconditional (or total) probability of an event B if you know its conditional probabilities with respect to some other event A and the probability of A . In this case, we knew directly what $P(B)$ is (because we had enough information- we know how many cards there are and how many spades) and you can see how it agrees with what the total law of probability gives you.

However, in most of the other examples, such as the one with the test for a virus we did in class, it's not possible to compute the probability of B (in that case, that the test is positive) directly because you don't have enough information (we don't know how many tests come out positive and how many tests are being administered, i.e., we don't know the percentage of tests that come out positive). What we know are the *conditional probabilities* of the test coming out positive with the conditions that the person taking it was infected or not. And we know the probability of this condition happening, i.e., we know the probability that someone is infected. So the information you have here consists of precisely the pieces that you need in order to use the total law of probability to compute the probability that a test comes out positive, and there's no other way to know this probability.

Remark 2. For all the following questions, the easiest way to think about them is to draw the tree diagram. Please do so when you try to do them, or when you read the solutions – draw the diagram to try to follow what's happening.

2. SOLUTIONS

Exercise 1. A doctor is called to see a sick child. The doctor has prior information that 90% of sick children in that neighborhood have the flu, while the other 10% are sick with

measles. Let F stand for an event of a child being sick with flu and M stand for an event of a child being sick with measles. Assume for simplicity that $F \cup M = \Omega$, i.e., that there no other maladies in that neighborhood.

A well-known symptom of measles is a rash (the event of having which we denote R). Assume that the probability of having a rash if one has measles is $P(R | M) = 0.95$. However, occasionally children with flu also develop rash, and the probability of having a rash if one has flu is $P(R | F) = 0.08$.

Upon examining the child, the doctor finds a rash. What is the probability that the child has measles?

Solution.

We use Bayes's formula.

$$\begin{aligned} P(M | R) &= \frac{P(R | M)P(M)}{(P(R | M)P(M) + P(R | F)P(F))} \\ &= \frac{0.95 \times 0.10}{(0.95 \times 0.10 + 0.08 \times 0.90)} \simeq 0.57. \end{aligned}$$

Which is nowhere close to 95% of $P(R|M)$.

Exercise 2. In a study, physicians were asked what the odds of breast cancer would be in a woman who was initially thought to have a 1% risk of cancer but who ended up with a positive mammogram result (a mammogram accurately classifies about 80% of cancerous tumors and 90% of benign tumors.)

95 out of a hundred physicians estimated the probability of cancer to be about 75%. Do you agree?

Solution.

Introduce the events:

$+$ = mammogram result is positive,

B = tumor is benign,

M = tumor is malignant.

Note that $B^c = M$. We are given $P(M) = .01$, so $P(B) = 1 - P(M) = .99$. We are also given the conditional probabilities $P(+ | M) = .80$ and $P(- | B) = .90$, where the event $-$ is the complement of $+$, thus $P(+ | B) = .10$

Bayes' formula in this case is

$$\begin{aligned} P(M | +) &= \frac{P(+ | M)P(M)}{(P(+ | M)P(M) + P(+ | B)P(B))} \\ &= \frac{0.80 \times 0.01}{(0.80 \times 0.01 + 0.10 \times 0.99)} \\ &\simeq 0.075 \end{aligned}$$

So the chance would be 7.5%. A far cry from a common estimate of 75

Exercise 3. Suppose we have 3 cards identical in form except that both sides of the first card are colored red, both sides of the second card are colored black, and one side of the third card is colored red and the other side is colored black.

The 3 cards are mixed up in a hat, and 1 card is randomly selected and put down on the ground. If the upper side of the chosen card is colored red, what is the probability that the other side is colored black?

Solution.

Let RR, BB, and RB denote, respectively, the events that the chosen card is the red-red, the black-black, or the red-black card. Letting R be the event that the upturned side of the chosen card is red, we have that the desired probability is obtained by

$$\begin{aligned} P(RB | R) &= \frac{P(RB \cap R)}{P(R)} \\ &= \frac{P(R | RB)P(RB)}{P(R | RR)P(RR) + P(R | RB)P(RB) + P(R | BB)P(BB)} \\ &= \frac{(\frac{1}{2})(\frac{1}{3})}{(1)(\frac{1}{3}) + (\frac{1}{2})(\frac{1}{3}) + 0(\frac{1}{3})} = \frac{1}{3} \end{aligned}$$

This question was actually just like the Monty Hall problem!

Exercise 4. It is estimated that 50% of emails are spam emails. Some software has been applied to filter these spam emails before they reach your inbox. A certain brand of software claims that it can detect 99% of spam emails, and the probability for a false positive (a non-spam email detected as spam) is 5%.

Now if an email is detected as spam, then what is the probability that it is in fact a non-spam email?

Solution.

Define events

A = event that an email is detected as spam,

B = event that an email is spam,

B^c = event that an email is not spam.

We know $P(B) = P(B^c) = .5$, $P(A | B) = 0.99$, $P(A | B^c) = 0.05$.

Hence by the Bayes's formula we have

$$\begin{aligned} P(B^c | A) &= \frac{P(A | B^c)P(B^c)}{P(A | B)P(B) + P(A | B^c)P(B^c)} \\ &= \frac{0.05 \times 0.5}{0.05 \times 0.5 + 0.99 \times 0.5} \\ &= \frac{5}{104}. \end{aligned}$$